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STRESSES IN A WEIGHABLE HALF-PLANE WITH A SEMICIRCULAR NOTCH

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The influence of the surface roughness on the stress state of a rock is studied. For an elastic half-plane in the gravity field that contains a notch shaped like a semicircle, the stress distribution is constructed. It is shown that depending on the Poisson ratio, the notch bottom can be in a state of tension or compression. The polynomial dependence of pressure on depth is given on the axis of symmetry.

The stress state of a geological section depends on the contour of seismic boundaries [1]. To analyze the stresses that are due to the surface roughness, we consider a half-plane weakened by a notch in the form of a semicircle.

1. There is a homogeneous elastic half-plane whose boundary has a notch in the form of a half-disk of unit radius in the gravity field. The equations of equilibrium have the form

$$\frac{\partial \sigma_{ik}}{\partial x_k} + \rho g_i = 0, \tag{1.1}$$

where ρ is the density and g_i is the component of the acceleration-of-gravity vector. The boundary conditions are the absence of loads at the boundary of the half-plane and the notch contour.

In the absence of a notch, the solution of the problem for a half-plane is written in the form

$$\sigma^0_{xx} = y\rho g\nu/(1-\nu), \qquad \sigma^0_{yy} = y\rho g, \qquad \sigma^0_{xy} = 0, \tag{1.2}$$

where ν is the Poisson ratio.

The general solution of (1.1) for a half-plane with a notch can be written as a sum of the particular solution (1.2) and the general solution of the homogeneous equation of equilibrium $\sigma_{ij} = \sigma^0_{ij} + \sigma^1_{ij}$. The additional stress field σ^1_{ij} should compensate for the loads $P_i = \sigma_{ij} n_j$ created by the particular solution (1.2) at the semicircle contour.

Assuming that the x axis coincides with the half-plane boundary and the y axis is directed upward, we have $n_x = -x$ and $n_y = -y$ on the semicircle of the projection of the normal vector; therefore, $P^0_x = -yx\rho g\nu/(1-\nu)$ and $P^0_y = -y^2\rho g$. On the complex plane, we have $x = (z+\bar z)/2$ and $y = (z-\bar z)/(2i)$; then

$$P^0_x + iP^0_y = -\frac{i\rho g}{4}\Big(2 - \frac{1}{1-\nu}\, z^2 - \frac{1-2\nu}{1-\nu}\, \bar z^2\Big), \qquad |z| = 1, \qquad \operatorname{Im} z < 0.$$

For the normal and tangential components of the load vector, which compensates for the particular solution (1.2) on the semicircle contour, we have $N - iT = (P^0_x - iP^0_y)z$. For σ^1_{ij}, the boundary condition on the semicircle contour is written in the form

$$N - iT = \frac{i\rho g}{4}\Big(2z - \frac{1}{1-\nu}\, \bar z - \frac{1-2\nu}{1-\nu}\, z^3\Big), \qquad |z| = 1, \qquad \operatorname{Im} z < 0. \tag{1.3}$$

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We now extend (1.3) to the region $\operatorname{Im} z > 0$. The resulting loads are antisymmetric with respect to the x axis.

2. We consider the unbounded plane with a cut circle of unit radius to which the loads (1.3) are applied. If (1.3) is represented in the form of a complex Fourier series

$$N - iT = \sum_{k=-\infty}^{+\infty} A_k e^{ik\theta}, \tag{2.1}$$

only the coefficients $A_1 = i\rho g/2$, $A_{-1} = -i\rho g/(4(1-\nu))$, and $A_3 = -i\rho g(1-2\nu)/(4(1-\nu))$ differ from zero.

Let $\Phi(z)$ and $\Psi(z)$ be functions that are homogeneous outside the circle $|z| = 1$; the expansion of these functions into a Laurent series has the form $\Phi(z) = \sum\limits_{k=0}^{\infty} a_k z^{-k}$ and $\Psi(z) = \sum\limits_{k=0}^{\infty} b_k z^{-k}$. According to [2], the coefficients a_k and b_k are related to the expansion coefficients (2.1) by the relations

$$a_0 = \Gamma, \qquad a_1 = \bar{A}_1/(1+æ), \qquad a_2 = \bar{\Gamma}' + \bar{A}_2, \qquad a_n = \bar{A}_n,$$

$$b_0 = \Gamma', \quad b_1 = -æA_1/(1+æ), \quad b_2 = 2\Gamma - A_0, \quad b_n = (n-1)a_{n-2} - A_{-n+2}, \quad n \geqslant 3,$$

where $æ = 3 - 4\nu$ in the case of plane strain and Γ and Γ' are the specified quantities which characterize the stress distribution at infinity.

In this case, only the coefficients

$$a_1 = -\frac{i\rho g}{8(1-\nu)}, \quad a_3 = \frac{i\rho g(1-2\nu)}{4(1-\nu)}, \quad b_1 = -\frac{i\rho g(3-4\nu)}{8(1-\nu)}, \quad b_5 = \frac{i\rho g(1-2\nu)}{1-\nu}$$

are not zero. Then, the potentials can be written in the form

$$\Phi(z) = -\frac{i\rho g}{8(1-\nu)}\Big(\frac{1}{z} - \frac{2(1-2\nu)}{z^3}\Big), \qquad \Psi(z) = -\frac{i\rho g}{8(1-\nu)}\Big(\frac{3-4\nu}{z} - \frac{8(1-2\nu)}{z^5}\Big).$$

Since the stress tensor is determined from the relations

$$\sigma_{xx} + \sigma_{yy} = 4\operatorname{Re}\Phi(z), \qquad \sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 2(\bar{z}\Phi'(z) + \Psi(z)),$$

we finally obtain

$$\sigma^1_{yy} = \frac{\rho g}{8(1-\nu)}\Big(-(5-4\nu)\frac{\sin\varphi}{r} + \frac{\sin 3\varphi}{r} + 4(1-2\nu)\frac{\sin 3\varphi}{r^3} - 6(1-2\nu)\frac{\sin 5\varphi}{r^3} + 8(1-2\nu)\frac{\sin 5\varphi}{r^5}\Big),$$

$$\sigma^1_{xy} = \frac{\rho g}{8(1-\nu)}\Big(\frac{\cos 3\varphi}{r} - (3-4\nu)\frac{\cos\varphi}{r} - 6(1-2\nu)\frac{\cos 5\varphi}{r^3} + 8(1-2\nu)\frac{\cos 5\varphi}{r^5}\Big), \tag{2.2}$$

$$\sigma^1_{xx} = \frac{\rho g}{8(1-\nu)}\Big((1-4\nu)\frac{\sin\varphi}{r} - \frac{\sin 3\varphi}{r} + 4(1-2\nu)\frac{\sin 3\varphi}{r^3} + 6(1-2\nu)\frac{\sin 5\varphi}{r^3} - 8(1-2\nu)\frac{\sin 5\varphi}{r^5}\Big),$$

where $r = \sqrt{x^2+y^2}$ and φ is the polar angle. For transition from the circle of unit radius to a circle of radius R, it is necessary to replace ρg in (2.2) by ρgR, and r by r/R.

It is easy to check that together with (1.2), the resulting solution (2.2) ensures zero loads on the circle contour. On the real axis, we have

$$\sigma^1_{xy} = -\frac{1-2\nu}{4(1-\nu)}\Big(\frac{1}{x} + \frac{3}{x^3} - \frac{4}{x^5}\Big)\rho g, \qquad \sigma^1_{xx} = \sigma^1_{yy} = 0. \tag{2.3}$$

Thus, solution (2.2) is not subject to the boundary conditions imposed on the additional field σ^1_{ij} on the real axis. We construct a solution σ^2_{ij} that ensures zero loads on the circle contour and tangential loads compensating for (2.3) on the real axis.

3. Let the concentrated force (P_x, P_y) be applied to the point $z_1 = (x_1, 0)$ $(|z_1| > 1)$ located on the real axis of the plane with a cut load-free circle of unit radius.

The potentials of the concentrated force applied at an arbitrary point z_1 of the unbounded plane have the form

$$\varphi(z) = -P\ln(z - z_1), \qquad \psi(z) = æ\bar{P}\ln(z - z_1) + \bar{z}_1 P/(z - z_1), \tag{3.1}$$

where $P = (P_x + iP_y)/b$ and $b = 2\pi(1+æ)$. If X and Y are the projections of the loads applied at the contour s, the limiting relation

$$\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} = i\int_0^z (X + iY)\,ds = f(z) \tag{3.2}$$

is satisfied on the contour [2]. Substituting the concentrated-force potentials (3.1) into (3.2), we have

$$f(\sigma) = -P\Big(\ln(\sigma - x_1) - æ\ln\Big(\frac{1}{\sigma} - x_1\Big)\Big) + \bar{P}\Big(\frac{x_1\sigma}{1 - x_1\sigma} - \frac{\sigma^2}{1 - x_1\sigma}\Big), \qquad |\sigma| = 1. \tag{3.3}$$

To unload the circle contour, we apply the loads $-f(\sigma)$ to it. We represent the solution for an unbounded plane with a cut circle whose contour is loaded in the form [2]

$$\varphi(z) = -\frac{1}{2\pi i}\int_{|\sigma|=1}\frac{f(\sigma)}{\sigma - z}\,d\sigma, \qquad \psi(z) = -\frac{1}{2\pi i}\int_{|\sigma|=1}\frac{\overline{f(\sigma)}}{\sigma - z}\,d\sigma - \frac{\varphi'(z)}{z}. \tag{3.4}$$

We write the conjugate boundary condition in the form

$$\overline{f(\sigma)} = -\bar{P}\Big(\ln\Big(\frac{1}{\sigma} - x_1\Big) - æ\ln(\sigma - x_1)\Big) + P\Big(\frac{x_1}{\sigma - x_1} - \frac{1}{\sigma(\sigma - x_1)}\Big), \qquad |\sigma| = 1.$$

We now substitute (3.3) and the resulting expression into (3.4). The integrals from (3.4) are calculated using the Cauchy theorem and integral formula. After calculating the integrals, we have

$$\begin{gathered}\varphi(z) = -Pæ\ln\Big(1 - \frac{1}{x_1 z}\Big) - \bar{P}\Big(\frac{1}{1 - x_1 z} - \frac{1}{x_1^2(1 - x_1 z)}\Big),\\ \psi(z) = \bar{P}\ln\Big(1 - \frac{1}{x_1 z}\Big) - P\frac{1}{x_1 z} - \frac{\varphi'(z)}{z}, \quad |z| > 1.\end{gathered} \tag{3.5}$$

Combining (3.1) and (3.5), we obtain the potentials that correspond to the concentrated force applied at a point on the real axis of the plane with a cut circle having a load-free contour:

$$\varphi(z) = W_1(z, x_1)P_x + iW_2(z, x_1)P_y, \qquad \psi(z) = W_3(z, x_1)P_x + iW_4(z, x_1)P_y, \tag{3.6}$$

where

$$bW_1(z, x_1) = -\ln(z - x_1) - æ\ln\Big(1 - \frac{1}{x_1 z}\Big) + \frac{1 - x_1^2}{x_1^2(1 - x_1 z)},$$

$$bW_2(z, x_1) = -\ln(z - x_1) - æ\ln\Big(1 - \frac{1}{x_1 z}\Big) - \frac{1 - x_1^2}{x_1^2(1 - x_1 z)},$$

$$bW_3(z, x_1) = æ\ln(z - x_1) + \frac{x_1}{z - x_1} + \ln\Big(1 - \frac{1}{x_1 z}\Big) - \frac{1 - x_1^2}{x_1 z(1 - x_1 z)^2} - \frac{æ}{z^2(1 - x_1 z)} - \frac{1}{x_1 z},$$

$$bW_4(z, x_1) = -æ\ln(z - x_1) + \frac{x_1}{z - x_1} - \ln\Big(1 - \frac{1}{x_1 z}\Big) + \frac{1 - x_1^2}{x_1 z(1 - x_1 z)^2} - \frac{æ}{z^2(1 - x_1 z)} - \frac{1}{x_1 z}.$$

On the abscissa axis, together with the concentrated load, the distributed load also corresponds to the potentials (3.6).

4. Let the continuous functions $p(t)$ and $\tau(t)$ be defined at the boundary L which consists of the rays $[-\infty, -1]$ and $[1, \infty]$ belonging to the real axis. Then the zero loads on the circle contour correspond to the potentials

$$\varphi(z) = \int_L (W_1(z,t)\tau(t) + iW_2(z,t)p(t))\,dt, \qquad \psi(z) = \int_L (W_3(z,t)\tau(t) + iW_4(z,t)p(t))\,dt,$$

which are homogeneous outside the unit circle. Using these potentials, we form the functional

$$\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} = \int_L (\Omega_1(z,t)\tau(t) + i\Omega_2(z,t)p(t))\,dt, \tag{4.1}$$

where

$$b\Omega_1(z,t) = -\ln(z-t) + æ\ln(\bar{z}-t) - æ\ln\Big(1-\frac{1}{tz}\Big) + \ln\Big(1-\frac{1}{t\bar{z}}\Big) - \frac{z-t}{\bar{z}-t}$$
$$+ \Big(z-\frac{1}{\bar{z}}\Big)\Big(\frac{1-t^2}{t(1-t\bar{z})^2} + \frac{æ}{\bar{z}(1-t\bar{z})}\Big) + \frac{1-t^2}{t^2(1-tz)} - \frac{1}{t\bar{z}},$$

$$b\Omega_2(z,t) = -\ln(z-t) + æ\ln(\bar{z}-t) - æ\ln\Big(1-\frac{1}{tz}\Big) + \ln\Big(1-\frac{1}{t\bar{z}}\Big) + \frac{z-t}{\bar{z}-t}$$
$$+ \Big(z-\frac{1}{\bar{z}}\Big)\Big(\frac{1-t^2}{t(1-t\bar{z})^2} - \frac{æ}{\bar{z}(1-t\bar{z})}\Big) - \frac{1-t^2}{t^2(1-tz)} + \frac{1}{t\bar{z}},$$

We differentiate (4.1) with respect to x on the real axis. By virtue of (3.2), for the left side of equality (4.1) we have

$$\frac{d}{dx}\,(\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}) = iX - Y. \tag{4.2}$$

The value of the integrand on the right side of (4.1) depends on from which half-plane the limiting passage to the real axis is carried out. In the form $\lim_{y\to\pm 0}(d\Omega_k(z,t)/dx)$, the terms containing the logarithmic function are multiple-value, because $\lim_{y\to 0}(d\ln(x \pm iy - t)/dx) = 1/(x-t) \mp i\pi\delta(x-t)$. The upper sign (plus or minus) corresponds to transition from the upper half-plane.

After the transition to the real axis, we obtain

$$b\frac{d}{dx}\,\Omega_i(x,t) = \pm i\pi(1+æ)\delta(x-t) - \frac{1-æ}{x-t} + K_i(x,t), \tag{4.3}$$

where

$$K_1(x,t) = \frac{1}{1-tx}\Big(\frac{2æ}{x^3} - \frac{1-æ}{x}\Big) + \frac{1}{tx^2} + \frac{2(1-t^2)}{(1-tx)^3}\Big(x-\frac{1}{x}\Big)$$
$$- \frac{1}{(1-tx)^2}\Big(2t - \frac{2}{t} - \frac{1}{tx^2} + \frac{t}{x^2} - æt + \frac{æt}{x^2}\Big),$$

$$K_2(x,t) = \frac{1}{1-tx}\Big(-\frac{2æ}{x^3} - \frac{1-æ}{x}\Big) - \frac{1}{tx^2} + \frac{2(1-t^2)}{(1-tx)^3}\Big(x-\frac{1}{x}\Big) + \frac{1}{(1-tx)^2}\Big(\frac{1}{tx^2} - \frac{t}{x^2} - æt + \frac{æt}{x^2}\Big).$$

After separation of the real and imaginary parts of the functions $p(t)$ and $\tau(t)$, from (4.1)–(4.3) we obtain the singular equations with Cauchy-type kernels:

$$\pm\pi(1+\text{æ})\tau(x)-(1-\text{æ})\int\limits_L \frac{p(t)}{x-t}\,dt+\int\limits_L K_2(x,t)p(t)\,dt=bX^{\pm},$$
$$\pm\pi(1+\text{æ})p(x)+(1-\text{æ})\int\limits_L \frac{\tau(t)}{x-t}\,dt-\int\limits_L K_1(x,t)\tau(t)\,dt=bY^{\pm}. \tag{4.4}$$

Thus, one can satisfy the given loads on L with the functions $p(t)$ and $\tau(t)$ (t) chosen properly.

5. If one sets æ $=1$ and $b=2\pi$ in (3.1), one obtains the potentials of the concentrated force applied to the half-plane boundary [2]. After this substitution, the potentials (3.6) can be treated as the potentials of the concentrated force applied to the boundary of a half-plane having a semicircle-shaped notch whose contour is free from loads. In this case, (4.4) is reduced to the form

$$\tau(x)-\int\limits_L K_2(x,t)p(t)\,dt=X,\qquad p(x)+\int\limits_L K_1(x,t)\tau(t)\,dt=Y, \tag{5.1}$$

where

$$2\pi K_1(x,t)=\frac{2}{(1-tx)x^3}+\frac{1}{tx^2}+\frac{2(1-t^2)}{(1-tx)^3}\Big(x-\frac{1}{x}\Big)+\frac{1}{(1-tx)^2}\Big(-t+\frac{2}{t}+\frac{1}{tx^2}-\frac{2t}{x^2}\Big),$$

$$2\pi K_2(x,t)=-\frac{2}{(1-tx)x^3}-\frac{1}{tx^2}+\frac{2(1-t^2)}{(1-tx)^3}\Big(x-\frac{1}{x}\Big)+\frac{1}{(1-tx)^2}\Big(\frac{1}{tx^2}-t\Big).$$

The sign of the load projections X and Y is changed, because the region remains on the right rather than on the left in the motion along the real axis in the positive direction. In contrast to (4.3), the kernels $K_1(x,t)$ and $K_2(x,t)$ include the factor 2π.

We now write the kernels of the resulting equations in the form of power series in x:

$$2\pi K_1(x,t)=\sum_{k=2}^{\infty}S_k^1x^{-k},\qquad 2\pi K_2(x,t)=\sum_{k=2}^{\infty}S_k^2x^{-k},$$

where $S_2^1=2t^{-1}$, $S_3^1=4t^{-2}-2t^{-4}$, $S_k^1=(k-1)(k-2)(-t^{-1}+2t-t^3)t^{-k}$, $S_2^2=-2t^{-3}$, $S_3^2=4t^{-2}-6t^{-4}$, and $S_k^2=(k-1)(-kt^{-1}+2(k-2)t-(k-4)t^3)t^{-k}$ for $k\geqslant 4$.

With allowance for the structures of the kernels and the equality

$$\int\limits_L t^{-n}\,dt=\begin{cases}2/(n-1), & n=0,2,4,\ldots,2k,\\ 0, & n=1,3,5,\ldots,2k+1,\end{cases}\qquad L=[-\infty,-1]\cup[1,\infty]$$

one can show that if X is an odd function and Y is an even function relative to x, we have $\tau(x)=-\tau(-x)$ and $p(x)=p(-x)$. We search for the densities of the potentials in the form

$$\tau(x)=\sum_{n=0}^{\infty}\frac{\alpha_{2n+1}}{x^{2n+1}},\qquad p(x)=\sum_{n=1}^{\infty}\frac{\beta_{2n}}{x^{2n}}. \tag{5.2}$$

Then, Eqs. (5.1) can be rewritten as follows:

$$X=\sum_{n=0}^{\infty}\frac{\alpha_{2n+1}}{x^{2n+1}}-\frac{1}{2\pi}\sum_{k=1}^{\infty}\frac{1}{x^{2k+1}}\sum_{m=0}^{\infty}\beta_{2m}\int\limits_L S_{2k+1}^2(t)t^{-2m}\,dt, \tag{5.3}$$

$$Y=\sum_{n=1}^{\infty}\frac{\beta_{2n}}{x^{2n}}+\frac{1}{2\pi}\sum_{k=2}^{\infty}\frac{1}{x^{2k}}\sum_{m=0}^{\infty}\alpha_{2m+1}\int\limits_L S_{2k}^1(t)t^{-2m-1}\,dt.$$

If X_{2k+1} and Y_{2k} are the expansion coefficients of the loads into series similar to (5.2), then, equating in (5.3) terms with the same powers x, we obtain the following infinite system of linear algebraic equations relative to α_k and β_k:

$$\alpha_{2k+1} - \sum_{m=1}^{\infty} \beta_{2m} B_{2m}^{2k+1} = X_{2k+1}, \qquad k = 0, 1, 2, \dots, \infty,$$
$$\beta_{2k} + \sum_{m=0}^{\infty} \alpha_{2m+1} A_{2m+1}^{2k} = Y_{2k}, \qquad k = 1, 2, 3, \dots, \infty, \tag{5.4}$$

Here

$$\pi A_{2m+1}^{2} = \frac{2}{2m+1}, \qquad B_{2m}^{1} = 0, \qquad \pi B_{2m}^{3} = \frac{4}{2m+1} - \frac{6}{2m+3},$$

$$\pi A_{2m+1}^{2k} = (2k-1)(2k-2)\Big(- \frac{1}{2m+2k+1} + \frac{2}{2m+2k-1} - \frac{1}{2m+2k-3}\Big),$$

$$\pi B_{2m}^{2k+1} = 2k\Big(- \frac{2k+1}{2m+2k+1} + \frac{2(2k-1)}{2m+2k-1} - \frac{2k-3}{2m+2k-3}\Big), \quad k \geqslant 2.$$

To compensate for the stresses (2.3) on L, we set the coefficients $X_1 = 1$, $X_3 = 3$, and $X_5 = -4$ different from zero on the right side of (5.4). Then, the coefficients α_k and β_k depend only on the index k, and the characteristics of the medium enter the solution (5.2) in the form of the factor $h = 0.25\rho g(1-2\nu)/(1-\nu)$.

If only the first M terms of expansion (5.2) are retained in the infinite system (5.4), its solution, for example, by the Gauss method with the choice of the principal element, has no difficulties. As $M \to \infty$, the solution converges, and, beginning with the fifth or sixth term, α_k and β_k decrease similarly to M^{-1}.

6. With α_k and β_k known, one can determine the stress state at each point of the half-plane. Here it is more convenient to pass to the potentials $\Phi(z) = \varphi'(z)$ and $\Psi(z) = \psi'(z)$:

$$\Phi(z) = \int_L (U_1(z,t)\tau(t) + iU_2(z,t)p(t))\, dt, \quad \Psi(z) = \int_L (U_3(z,t)\tau(t) + iU_4(z,t)p(t))\, dt. \tag{6.1}$$

Here

$$2\pi U_1(z,t) = \frac{1}{t-z} + \frac{1}{z(1-zt)} + \frac{1-t^2}{t(1-zt)^2}, \qquad 2\pi U_2(z,t) = \frac{1}{t-z} + \frac{1}{z(1-zt)} - \frac{1-t^2}{t(1-zt)^2},$$

$$2\pi U_3(z,t) = -\frac{1}{t-z} - \frac{t}{(z-t)^2} - \frac{1}{z(1-zt)}$$
$$+ \frac{2}{z^3(1-tz)} + \frac{1-t^2}{tz^2(1-tz)^2} - \frac{t}{z^2(1-tz)^2} - \frac{2(1-t^2)}{z(1-tz)^3} + \frac{1}{tz^2},$$

$$2\pi U_4(z,t) = \frac{1}{t-z} - \frac{t}{(z-t)^2} + \frac{1}{z(1-zt)} + \frac{2}{z^3(1-tz)} - \frac{1}{tz^2(1-tz)^2} + \frac{2(1-t^2)}{z(1-tz)^3} + \frac{1}{tz^2}.$$

We define the following integrals [3]:

$$I_{2n+1} = \int_L \frac{dt}{t^{2n+1}(t-z)} = \frac{1}{z^{2n+1}}\Big(\ln\Big(\frac{z+1}{z-1}\Big) - i\pi\Big) - \sum_{k=1}^{n} \frac{2}{(2n-2k+1)z^{2k}},$$

$$I_{2n} = \int_L \frac{dt}{t^{2n}(t-z)} = \frac{1}{z^{2n}}\Big(\ln\Big(\frac{z+1}{z-1}\Big) - i\pi\Big) - \sum_{k=1}^{n} \frac{2}{(2n-2k+1)z^{2k-1}},$$

$$J_{2n+1} = \int\limits_L \frac{dt}{t^{2n+1}(1-zt)} = -z^{2n} \ln\left(\frac{z+1}{z-1}\right) + \sum_{k=1}^{n} \frac{2z^{2k-1}}{2n-2k+1},$$

$$J_{2n} = \int\limits_L \frac{dt}{t^{2n}(1-zt)} = -z^{2n-1} \ln\left(\frac{z+1}{z-1}\right) + \sum_{k=1}^{n} \frac{2z^{2k-2}}{2n-2k+1}, \qquad n = 0, 1, 2, \ldots, \infty,$$

$$R_0 = 2/(z^2-1), \quad Q_0 = -2/(z^2-1), \quad H_0 = -2/(z^2-1)^2, \quad H_{-1} = -2z/(z^2-1)^2, \tag{6.2}$$

$$R_n = \int\limits_L \frac{dt}{t^n(1-zt)^2} = nz^{n-1}\left(\frac{2z}{z^2-1} - \ln\left(\frac{z+1}{z-1}\right)\right)$$

$$-\sum_{k=1}^{n-1} \frac{nz^{k-1}}{(n-k)(n-k+1)}\left(\frac{(-1)^{n-k}}{1+z} - \frac{1}{1-z}\right),$$

$$Q_n = \int\limits_L \frac{dt}{t^n(z-t)^2} = -\frac{n}{z^n}\left(\frac{2}{z^2-1} + \frac{1}{z}\left(\ln\left(\frac{z+1}{z-1}\right) - i\pi\right)\right)$$

$$-\sum_{k=1}^{n-1} \frac{n}{(n-k)(n-k+1)z^k}\left(\frac{(-1)^{n-k}}{z+1} - \frac{1}{z-1}\right),$$

$$H_n = \int\limits_L \frac{dt}{t^n(1-zt)^3} = \frac{n(n+1)z^{n-1}}{2}\left(\frac{2z(z^2-2)}{(z^2-1)^2} - \ln\left(\frac{z+1}{z-1}\right)\right)$$

$$-\sum_{k=1}^{n-1} \frac{n(n+1)(n-k)^{-1}z^{k-1}}{(n-k+1)(n-k+2)}\left(\frac{(-1)^{n-k}}{(1+z)^2} - \frac{1}{(1-z)^2}\right), \quad n = 1, 2, \ldots, \infty.$$

The expressions for J_n, R_n, and H_n contain the differences of the power functions z^n. For large z and n, the calculations in finite-valued arithmetic become impossible because of rapidly growing errors. Therefore, for $z > 1$, we use the expressions derived from (6.2) by means of series expansions:

$$J_{2n+1} = -2\sum_{k=0}^{\infty} \frac{z^{-2k-1}}{2n+2k+1}, \qquad J_{2n} = -2\sum_{k=0}^{\infty} \frac{z^{-2k-2}}{2n+2k+1},$$

$$R_{2n+1} = 2\sum_{k=0}^{\infty} \frac{(2k+2)z^{-2k-3}}{2n+2k+3}, \qquad R_{2n} = 2\sum_{k=0}^{\infty} \frac{(2k+1)z^{-2k-2}}{2n+2k+1},$$

$$H_{2n+1} = -\sum_{k=0}^{\infty} \frac{(2k+1)(2k+2)z^{-2k-3}}{2n+2k+3}, \qquad H_{2n} = -\sum_{k=0}^{\infty} \frac{(2k+2)(2k+3)z^{-2k-4}}{2n+2k+3}.$$

Substituting (5.2) into (6.1) and taking into account (6.2), we obtain the desired potentials

$$\frac{2\pi}{h}\,\Phi(z) = \sum_{n=0}^{\infty} \alpha_{2n+1}\Big(I_{2n+1} + \frac{J_{2n+1}}{z} + R_{2n+2} - R_{2n}\Big) + i\sum_{n=1}^{\infty} \beta_{2n}\Big(I_{2n} + \frac{J_{2n}}{z} - R_{2n+1} + R_{2n-1}\Big),$$

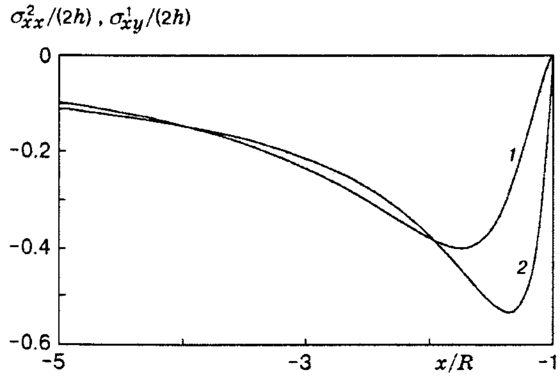


Fig. 1

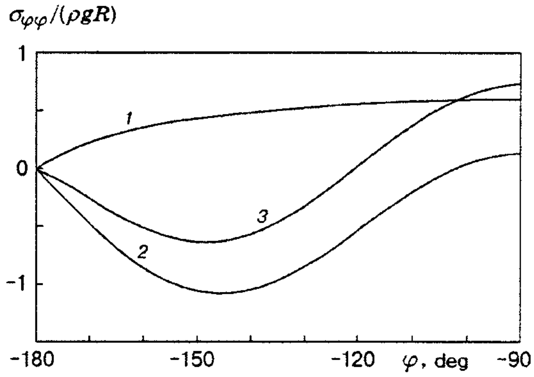


Fig. 2

$$\frac{2\pi}{h}\,\Psi(z) = i\sum_{n=1}^{\infty}\beta_{2n}\Big(I_{2n} - Q_{2n-1} + \Big(\frac{1}{z} + \frac{2}{z^3}\Big)J_{2n} - \frac{R_{2n+1}}{z^2} + \frac{2}{z}\,(H_{2n} - H_{2n-2})\Big) \tag{6.3}$$

$$+\sum_{n=0}\alpha_{2n+1}\Big(-I_{2n+1} - Q_{2n} - \Big(\frac{1}{z} - \frac{2}{z^3}\Big)J_{2n+1} + \frac{1}{z^2}\,(R_{2n+2} - 2R_{2n}) - \frac{2}{z}\,(H_{2n+1} - H_{2n-1}) + \frac{2}{z^2(2n+1)}\Big).$$

Thus, the stresses in an elastic half-plane with a notch in the form of a half-disk in the gravity field are the sum of stresses (1.2), (2.2), and (6.3). From the symmetry of the problem, it follows that σ_{xx} and σ_{yy} are even functions relative to x, and σ_{xy} is an odd function.

7. We represent the total stresses in the form $\sigma_{ij} = \sigma_{ij}^0 + \sigma_{ij}^1 + \sigma_{ij}^2$, where σ_{ij}^2 are the stresses calculated from (6.3). The calculations were performed for a notch of radius R; the coordinates are referred to the radius, and the stresses to $\rho g R$.

Figure 1 shows the stresses σ_{xx}^2 (curve 1) and σ_{xy}^1 (curve 2) at the boundary $y = 0$. The stresses are referred to $2h$ [here $2h = 0.5\rho g R(1-2\nu)/(1-\nu)$] and do not depend on the characteristics of the medium. By virtue of the boundary conditions, the stresses σ_{yy}^2 are zero, and $\sigma_{xy}^2(x,0) = -\sigma_{xy}^1(x,0)$. With allowance for (1.2) and (2.2), we find that curve 1 in Fig. 1 corresponds to the single nonzero component of the total-stress tensor at this boundary.

The error of the calculations by means of (6.3) is connected mainly with the fact that we should confine ourselves to a finite number of desired quantities in the solution of the infinite system (5.4). However, this error becomes noticeable only in the vicinity of the angular points $(\pm R, 0)$.

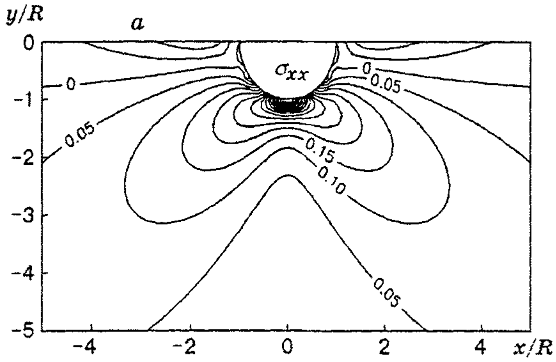
In the calculations, 200 terms were kept in system (5.4). Beginning with $M = 25$, the stress distributions do not as a matter of fact depend on the number of retained terms, except for the component σ_{xx}^2 in the interval $-1.32R \leqslant x \leqslant -R$. As the dimensionality of the system grows, the solution converges slowly to the value of $\sigma_{xx}^2(-R, 0) = 0$, which follows from the boundary conditions on the semicircle. The polynomial interpolation was performed in the indicated interval.

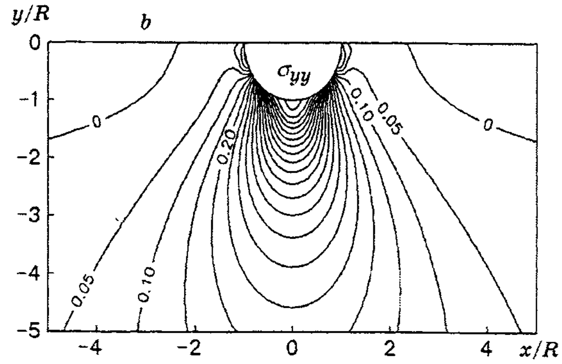
The material near the boundary is in a state of compression, because $\sigma_{kk} < 0$. For $x \approx -1.74R$, the compression reaches the maximum values of $\sigma_{xx} \approx -0.2\rho g R(1-2\nu)/(1-\nu)$. If $x \to \infty$, we have $\sigma_{xx} \sim -(R/x)^2$.

In the solution, the quantity σ_{ij}^2 and the sum of σ_{ij}^0 and σ_{ij}^1 give zero loads on the semicircle contour; therefore, in the polar coordinate system, only the stress-tensor component $\sigma_{\varphi\varphi}$ is different from zero; the dependence of this component on the polar angle φ is shown in Fig. 2 [curve 1 refers to $\sigma_{\varphi\varphi}^2$, curve 2 to $\sigma_{\varphi\varphi}^0 + \sigma_{\varphi\varphi}^1$, and curve 3 to the sum of the stresses (1.2), (2.2), and (6.3)].

In the calculations, the Poisson ratio was assumed to be equal to 0.2. For a $\pm 30°$ deviation from the vertical, the notch contour is in a state of tension, and the other sections of the contour are in a state of compression.

At the notch bottom, at the point $(0, -R)$ the solution (6.3) gives a single nonzero component of the





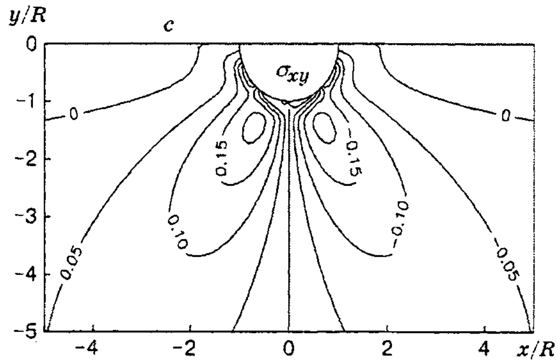


Fig. 3

stress tensor $\sigma_{xx}^2/(2h) = 1.59$. For the total-stress tensor [the sum (1.2), (2.2), and (6.3)], we can write

$$\sigma_{xx} = 0.5\rho gR((1-4\nu) + 1.59(1-2\nu))/(1-\nu), \qquad \sigma_{yy} = \sigma_{xy} = 0.$$

The stress concentration at this point reaches a significant magnitude and is determined by the Poisson ratio. The notch bottom can be in a state of tension ($\sigma_{xx} = 1.04\rho gR$ for $\nu = 0.1$) or compression ($\sigma_{xx} = -0.235\rho gR$ for $\nu = 0.4$). The change of the types of stress occurs for $\nu = 0.361$.

The average normal stress has the form $P = -(\sigma_{xx} + \sigma_{yy})/2$. Applying the polynomial approximation to the calculation results obtained from (6.3) and taking into account the contribution from stresses (1.2) and (2.2), we represent the average normal stresses on the straight line $x = 0$ in the form

$$P(\xi) = -\frac{\rho gR}{2(1-\nu)}\Big(\xi - \frac{1}{2\xi} - \frac{1-2\nu}{\xi^3} - (1-2\nu)\Big(\frac{0.494}{\xi} - \frac{1.301}{\xi^2} - \frac{2.239}{\xi^3} - \frac{2.133}{\xi^4} - \frac{1.194}{\xi^5} - \frac{0.3}{\xi^6}\Big)\Big),$$

where $\xi = y/R$. For $\nu < 0.361$, in the vicinity of the notch bottom there is a zone of tensile stresses in which $P < 0$. The depth of this zone is quite significant; it reaches $-1.275R$ for $\nu = 0.1$ and $-1.135R$ for $\nu = 0.25$.

The stress (1.2) increases by a linear law with depth. An additional stress field $(\sigma_{ij}^1 + \sigma_{ij}^2)$, which decreases with distance from the notch not slower than r^{-1}, is imposed on (1.2). Figure 3a–c shows isolines of the additional field of stresses σ_{xx}, σ_{yy}, and σ_{xy}. The calculations were performed for $\nu = 0.2$. As before, the stresses are referred to ρgR.

In the narrow zones adjacent to the boundary $y = 0$, compression occurs in the horizontal direction $(\sigma_{xx}^1 + \sigma_{xx}^2 < 0)$, and expansion occurs on the other part of the half-plane $(\sigma_{xx}^1 + \sigma_{xx}^2 > 0)$ (Fig. 3a). The influence of the notch decreases rapidly with distance from it. The maximum stresses σ_{xx} are reached at the notch bottom and on the rays located at the angles $\pi/4$ and $-\pi/4$ to the coordinate axes.

The stresses σ_{yy} of the additional field (Fig. 3b) are localized under the notch and, as a result, the material extends in the vertical direction $(\sigma_{yy}^1 + \sigma_{yy}^2 > 0)$. The maximum value of σ_{yy} is attained at the notch bottom and is equal to ρgR according to the boundary conditions.

The petals of the isolines of the stresses σ_{xy} of the additional field are elongated at the angles $\pi/3$ and $-\pi/3$ (Fig. 3c). The maximum value of $\sigma_{xy} = 0.28\rho g R$ is reached at the points with coordinates ($\pm 0.7R$; $-1.46R$) rather than on the notch contour.

Thus, the notch is a concentrator of stresses. The effect of the notch on the stress state of rocks is local; at the depth $5R$, the contribution to the average normal stresses is 3.3%, which is 5 MPa, for example, for $\rho = 2.55$ g/cm^3 and $R = 200$ m. These gradients can cause a redistribution of fluids that saturate productive layers, thus creating potential traps of the kind for them.

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